On dimensional analysis

On dimensional analysis

J C Gibbings

Fluid Mechanics Division, University of Liverpool, PO Box 147, Liverpool L69 3BX, UK

Abstract. Comments are made upon aspects of dimensional analysis illustrating matters such as the distinction in a definition between a quantity and its measure and the effect upon formulation of the analysis of approximate modelling, of the uncoupling of equations and of the inclusion of thermodynamic properties as variables. The status of angle and of amount of substance as primary quantities is discussed. A treatment of unit conversion factors is given.

1. Introduction

The literature of dimensions and units is voluminous, though much of it excludes or barely touches upon dimensional analysis which is the prime concern of this paper. The more elementary treatments of dimensional analysis too frequently can be faulted, whilst the specialised monographs often leave questions unanswered. Detailed scrutiny of many expositions makes it difficult to advance a defence against the commonly made claim that dimensional analysis is only effective because the correct answer has previously been otherwise obtained. When the position has been reached of publication of an unnecessarily complicated derivation which in addition is not soundly based, arriving at an answer that is incorrect, then the time has come to reconsider; this is what this paper sets out to do.

But that is not all; as Jeffreys pointed out (Jeffreys 1943, see p 839, LI 22–5), too often difficulties arise because of semantic problems. On other occasions difficulties can be due to the use of definitions that are lacking in rigour, such as confusion between the use of mass in a momentum context and in a thermal internal-energy context as a unit of amount of substance. Or again, and this is most common in philosophical writings (Plotinus 1968 (Time and Eternity), Waismann 1968 (Analytic-synthetic)), difficulties have arisen because the careful distinction , necessary in physics, between the definition of the nature of a physical quantity and the definition of its measure is not made. Definitions making this distinction are given here.

In this paper the following points are considered:
(a) The fundamental definition of basic quantities such as length and time.
(b) The necessity of assigning dimensions to angle and to amount of substance.
(c) The units of certain molecular quantities and the corresponding unit conversion factors.
(d) The choice of variables based upon the physics of a problem and their total number as required by the tenets of thermodynamics.
(e) The questionable use of directional concepts in dimensional analysis and the direct relevance of the uncoupling of governing equations.

(f) The Rayleigh–Riabouchinsky problem.

(g) The relation between the number of base quantities and the number of unit conversion factors.

2. The definitions of physical quantities

It has been mentioned how a difficulty has arisen amongst philosophers which appears to be due to lack of distinction between the definition of a quantity and the definition of its means of measurement. There is a difference in approach between the 'philosopher' and the physicist because the latter is faced with the reality of having to measure. A particular example arises in the definition of time: on the basis that, here, science can aid philosophy, this is now discussed.

Consider as a preliminary the elementary concept of extension†. We can appreciate position by our senses; we can observe the absence of sameness in position between two positions. This difference is defined as the concept of extension. The definition of its numerical measure must then be a consequence‡; it is measured by counting a set of standard lengths. The precision is controlled by the size of the smallest standard length; in the language of physics, by the degree of division of the standard metre.

This primary definition of extension must embrace the requirements of the secondary definition of length measurement. That is why in this definition of extension one has to recognise that objects must be placed to terminate an extension; the present 'absence of sameness in position' must be in relation to objects occupying the two positions. Then a 'position' in empty space is an invaluable analytical abstraction as is 'infinity' or a 'point'. Again the relation between two objects that measures a length can be determined by relating each of these objects to a third§; all physical measurement is a matter of comparison, and the observer, having made the measurement, is not concerned that it would change if he walked away.

Amongst philosophers difficulty has existed over the definition of time. Plotinus (1968 (Time and eternity)) criticised Aristotle's definition as being 'circular' in that it relied on the definition of motion‖ which in turn relied on the definition of time. This difficulty is still discussed (Waismann 1968, (Analytic-synthetic)). However, an approach analogous to that just given for extension and for length seems fruitful.

For, similarly, we can appreciate an accumulation of experience; then we can appreciate an addition of experience. Thus we can appreciate a difference in a total of experience. This difference can now be defined as time; it is not then a definition linked with motion.

For amplification, consider that an experimenter 'A' recognises a 'point' in time by observing one or more events, indicated by '*a*', up to that time, and others after it. Then experimenter 'B' recognises the same point in time by observing '*b*' which may contain some or all of '*a*' so that 'A' and 'B' can agree on this 'point' in time. Then experimenter 'C' recognises the same 'point' in time by observing '*c*'. If '*c*' contains some or all of '*b*'

---

† A physicist usually calls this 'length', whereas the philosopher uses the word 'length' to denote the numerical measure.

‡ This approach is the reverse of that proposed by Taylor (1974, p 2) who says that 'it is important that the concept . . . as determined from the rules for its measurement . . .'.

§ Here there is an analogy with the zeroth law of thermodynamics (Gibbings 1970).

‖ It seems that some philosophers use motion and time as synonyms; to a physicist, motion is the relation between space and time measured, for example, as a velocity.

then 'B' and 'C' agree on the location of the 'point' in time; 'c' may contain none of 'a' but still 'C' and 'A' can agree†. The present definition seems to overcome further philosophical difficulties concerning the zero extent of 'past', 'present' and 'future' (Augustine 1961)‡. For if 'present' is regarded as synonymous with a prescribed 'point' in time, it becomes the useful analytical abstraction that a 'point' in extension is. As before, the numerical measure of time then follows; in physics it is measured by counting what are assumed to be identical events.

Another definition is worth recalling; it is that for mass given in Gibbings (1970). Unlike those above for extension and time it is analogous to those for other physical properties such as viscosity, conductivity and so on (Gibbings 1970, ch 13). In contrast, the above definitions for extension and time are analogous to that for internal energy (Gibbings 1970, pp 70–1, Keenan 1957).

3. The use of units conversion factors

Dimensional analysis is founded on the principle of equality of the dimensional structure of those terms in an equation that are arithmetically summed. A use of dimensions that are related by units conversion factors requires the appearance of these factors in the equations. This would suggest that these factors should also appear as variables in a formulation of a dimensional analysis: problems arise over the necessity and choice of these factors.

The definition of measure of an area a is the arithmetic sum of unit squares: then we say that the area of a square is given by the square of the side, l, or $a = l^2$. But, in general, we should write

$$a = A_0 l^2. \tag{1}$$

To explain the nomenclature used here consider two cases:
    (a) where a is measured in acres and l is measured in yards, then:
        (i) A_0 has units i.e. acre $(\text{yard})^{-2}$;
        (ii) A_0 has dimensions i.e. area $(\text{length})^{-2}$;
        (iii) A_0 has the numerical value of 1/4840.
    (b) where a is measured in metre^2 and l is measured in metres, then:
        (i) A_0 has no units;
        (ii) A_0 has no dimensions;
        (iii) A_0 has the numerical value of 1·0.
Jeffreys has pointed out in a similar context (Jeffreys 1943, p 840) that we could alternatively fix the dimensions of l and of A_0 so that those of a would follow. Or, again, the numerical value of A_0 can be specified; an example of this arises in magnetism, where, in the SI system, the value $4\pi 10^{-7}$ is assigned to μ_0.

It is seen that A_0 is a units conversion factor. The procedure (b) just described is commonly adopted, but, because of the assignment of a numerical value of unity, it can readily go unrecognised. As an example where difficulty has arisen consider the quantity 'angle'.

---

† Again there is an analogy with the statement of the zeroth law of thermodynamics.
‡ Saint Augustine poses the problem, asking '... the present, which we found was the only one of the three divisions of time (past, present, future) that could possibly be said to be long ...' and '... the present ... has no duration'.

Angle is a unique quantity in that the international standard is, in principle, absolutely accurate. Calling that standard angle any one of 360°, 400 grad or 2π radians is a matter of units†. Then, as the numerical measure of an angle depends on the size of the basic unit, it must have a dimension. Therefore, as for linear motion

$$F = g_0 m (d^2 s / dt^2),$$

where g_0 is a units conversion factor, then so for angular motion,

$$F . l = \beta_0 g_0 I (d^2 \beta / dt^2)$$

where β_0 is the units conversion factor for angle. A defining relation for β_0 would then be

$$s/r = \beta_0 \alpha,$$

where s and r are respectively the perimeter and radius of a sector of a circle subtending the angle α.

The necessity for the introduction of β_0 into dimensional analysis can be seen in Taylor's example of the loading of a frame. Taylor puts the deflection, δ, as a function of the load, W, upon a triangular frame whose shape is controlled by two angles of the triangle, β and γ, and whose size is measured by the length of one side l. The further variables are the cross-sectional area of the members, A, and Young's modulus, E. Then, assigning a dimension to angle, Taylor obtained

$$\phi\left(\frac{\delta}{l}, \frac{\beta}{\gamma}, \frac{W}{l^2 E}, \frac{A}{l^2}\right) = 0$$

which, as he pointed out, is incorrect. By introducing the units conversion factor, β_0, the extra group $\beta_0 \beta$ necessary to resolve Taylor's difficulty is obtained. We conclude that angle must be assigned a dimension, with its appropriate units conversion factor, so that, for the purposes of dimensional analysis, there is no reason for this quantity to have a subsidiary ranking in a system of units such as in the SI one‡.

Another quantity is now considered in the same way; it is the amount of substance measured in moles. This molar quantity is introduced from the relations for gases; those that have to be accounted for in a dimensional analysis are:

$$p/\rho = (R/M_0) T, \tag{2}$$

$$m = M_0 A_m, \tag{3}$$

$$A_m = 1/N_a, \tag{4}$$

$$k_B = R/N_a, \tag{5}$$

$$\tfrac{1}{2} m \bar{c}^2 = \tfrac{3}{2} k_B T. \tag{6}$$

By assigning $[m] = M$ then§ the dimensions of k_B are, from equation (6),

$$[k_B] = ML^2 T^{-2} \theta^{-1}. \tag{7}$$

---

† Philosophically, the last of these is the least preferable as indeterminancy of the unit is fixed by the indeterminancy of π.

‡ There has been a proposal that the measure of area should involve that of angle (Page 1967). Our preference for its definition is given at the beginning of this section.

§ We use Taylor's notation that [] = means 'is dimensionally identical'.

Also, from equation (2),

$$[R/M_0] = L^2T^{-2}\theta^{-1}. \tag{8}$$

From equations (3), (4) and (5),

$$k_B/m = R/M_0, \tag{9}$$

making these equations dimensionally consistent with equations (7) and (8).

If the amount of substance is introduced as a fundamental quantity, and denoted by n, then assigning

$$[N_a] = n^{-1} \tag{10}$$

as required by the physical concept of Avogadro's number and by the definition of amount of substance, it follows that from equation (4)

$$[A_m] = n,$$

so that the so-called unit atomic mass does not have the dimensions of mass per unit reference particle. Further, the molecular mass has, from equation (3), the dimensions

$$[M_0] = Mn^{-1}, \tag{11}$$

so that it is not purely a numerical factor; it has dimensions. Then, from equation (5),

$$[R] = ML^2T^{-2}\theta^{-1}n^{-1}. \tag{12}$$

An illustration of the need of the introduction of the amount of substance comes from the following example of the use of dimensional analysis to derive the expression, given by the kinetic theory of gases, for the thermal conductivity λ.

In the first instance, one writes

$$\lambda = \lambda(N, l_m, \bar{c}, m), \tag{13}$$

where N is the number of molecules per volume unit, l_m is the molecular mean free-path, $\bar{c}$ the mean speed, and m the mass of a molecule as before. However, the continuum definition of λ involves temperature as a unit, so that a units conversion factor has to be introduced into equation (13). Equation (2) is the defining relation for the numerical scale of temperature. Then, if the appropriate units conversion factor is chosen as R, if no account is taken of the dimensions of molar quantity, and if M_0 is assigned zero dimensions, successive use of equations (3), (4) and (5) gives $[R] = L^2T^{-2}\theta^{-1}$. Consequently the incorrect result is given by dimensional analysis as[†]

$$\lambda/mR\bar{c}Nl_m = \text{constant},$$

which by equation (9) is equivalent to

$$\lambda/M_0k_B\bar{c}Nl_m = \text{constant}.$$

The correct result from kinetic theory is

$$\lambda/k_B\bar{c}Nl_m = \text{constant}. \tag{14}$$

If the amount of substance is introduced as an extra primary quantity and the molecular

---

† The step whereby two Π groups are reduced to this one is discussed in another manuscript, but is basically a physical supposition that, other things being fixed, $\lambda \propto N$.

mass M_0 as an extra variable into equation (13), a single group is obtained again, now as

$$\lambda M_0 / m R \bar{c} N l_m = \text{constant},$$

which by equation (9) is equation (14), the correct result.

Equation (14) can be obtained directly if, noting equation (6), k_B is added to equation (13) as the units conversion factor for temperature rather than R. This could be advantageous in avoiding the need for the use of the amount of substance required in this example. There is another reason shown by this example for preferring k_B as a measure of the units conversion factor rather than R; for the latter is then introduced together with M_0 and m as variables, whereas equation (3) shows that the last two are not independent. Equation (14) could also be obtained if the quantity R/M_0 is taken as the units conversion factor for temperature, but this is not considered a suitable proposal because R/M_0 is not a universal constant.

Thus k_B can be adopted as the units conversion factor for temperature and subsequently R becomes the units conversion factor for amount of substance. Consequently, use of R in analysis is interdependent on use of amount of substance as a primary quantity.

4. Choice of variables

There are examples in the literature that give an appearance of relying, for the choice of variables, more upon a prior knowledge of the answer than upon a basis of physics. This fault appears for two reasons. One is a failure to state the analytical approximations that control the choice of independent variables, and the other is in choosing variables that are not obvious from a first assessment of the physics of a phenomenon. The following examples illustrate this.

The first is the derivation of the relation for the force on a body immersed in a steady uniform stream where the fluid is incompressible. In specifying the variables of the problem the approximation inherent in the governing equations must be observed. These variables are p, ρ, μ, g, and the velocity U, whilst the size of the system is indicated by l. The approximations imply that ρ and μ are constant throughout the flow and equal to their boundary values. The other boundary conditions are p_0 and U whilst p and F are alternative dependent variables. Thus the force can be expressed as

$$F = F(p_0, \rho, \mu, U, l, g), \tag{15}$$

which leads to

$$\frac{F}{\rho U^2 l^2} = f\left(\frac{p_0}{\rho U^2}, \frac{\rho U l}{\mu}, \frac{U^2}{gl}\right).$$

Considering the first independent non-dimensional group, this must be included for the correct general result; the streamwise force on the external surface of a gas turbine nacelle is a function of the absolute pressure when the inlet and outlet areas are unequal. The independence of absolute pressure is generally valid only for a closed body. To demonstrate this we must refer to the physics. In the governing momentum equations (Sabersky and Acosta 1964), the pressure appears only as a gradient and so the velocity distribution is independent of absolute pressure p_0, and then so also is the distribution of shear stress; for a closed body, the surface pressure integral for the force is then independent of the absolute pressure because over this closed surface the

appropriate integral of p_0 is zero. Only then can this first non-dimensional group be excluded.

The third non-dimensional group is a measure of the effects of weight force. But, such are the governing equations that in them the effects of gravity can be incorporated into the pressure term (Sabersky and Acosta 1964), so that this non-dimensional group can also be excluded; but there is a proviso. If the boundary conditions are not fixed, so that, as with a free-surface to a liquid, the boundary shape is itself a function of the imposed flow, then such boundary conditions require the retention of the third non-dimensional group.

Hence the present thesis is that a final reduction of equation (15) to

$$\frac{F}{\rho U^2 l^2} = f\left(\frac{\rho U l}{\mu}\right)$$

requires the foregoing discussion of its limitations; it is not acceptable merely to state *a priori* that

$$F = F(\rho, \mu, U, l).$$

The second reason given at the beginning of this section is illustrated by the following example. It is the incompressible flow along a pipe of diameter d and length L, for which the pressure change, Δp, is given by

$$\Delta p = F(\rho, \mu, U_0, d, L),$$

where U_0 is the velocity on the centre-line of the pipe. Thus

$$\frac{\Delta p}{\rho U_0^2} = f\left(\frac{\rho U_0 d}{\mu}, \frac{L}{d}\right).$$

For that portion of the flow well away from the entry to the pipe, physical arguments show that Δp will be proportional to L. Thus

$$\frac{\Delta p \cdot d}{\rho U_0^2 L} = f\left(\frac{\rho U_0 d}{\mu}\right). \tag{16}$$

A further limitation is to specify laminar flow in which all the streamlines are then straight and parallel and so all particles of fluid are in unaccelerated motion. Thus density can be excluded as a variable, so that equation (16) reduces to

$$\frac{\Delta p}{L} \cdot \frac{d^2}{\mu U_0} = \text{constant}. \tag{17}$$

If there is interest in the rate of mass flow through the pipe, it is recognised that the mass is then used to represent a quantity of material. This is a far more common practice amongst engineers than chemists as the latter make use of a quantity measure in units of the mole. So, considering a volume flow rate Q, the density ρ can still be excluded, giving

$$Q = f(U_0, d, \mu)$$

or

$$Q/U_0 d^2 = \text{constant},$$

and μ does not appear. If required, the velocity U_0 can be eliminated by equation (17) to give

$$\frac{\Delta p}{L} \cdot \frac{d^4}{\mu Q} = \text{constant.}$$

As the mass flow rate, $\dot{m}$, is equal to ρQ, this latter equation can be rewritten as

$$\frac{\Delta p}{L} \cdot \frac{\rho d^4}{\mu \dot{m}} = \text{constant.}$$

Difficulties that have arisen in the literature in deriving this relation by inclusion of $\dot{m}$ in the original list of variables would not arise were the amount of substance rather than the mass to be used. This seems a clearer approach than the concept that has been advanced in the past of using a second unit of mass for the measurement of quantity of material.

5. Vectorial equality

The previous example was given for another reason. It has been derived in the literature by purporting directional characteristics to certain variables. But this, like many other similarly treated examples (Gibbings 1974), can be derived without ascribing such characteristics. Some authors have allocated directional characteristics to lengths where the normal vector concept is not applicable (Guggenheim 1942, p 495). But some such examples as have been proposed are open to question; even the simple one by Guggenheim concerning railway lines (tracks) is not acceptable were one to be interested in the total length of wood comprising the sleepers (ties).

The motion of a projectile under gravity is an example where, commonly, authors (e.g. Huntley 1952) have proposed the use of more than one length dimension to take account of directional characteristics. What has to be recognised in this problem is that the equations of motion in the vertical and horizontal directions are uncoupled; that is, they do not have to be solved simultaneously but can be solved successively for the two unknowns, t the time and then R the range of flight: this then directly effects the choice of what are truly independent variables. So, for the vertical motion[†],

$$t = t(h, g, v_0),$$

giving

$$\frac{t^2 g}{h} = f\left(\frac{v_0^2}{hg}\right).$$

Then for $v_0 = 0$, $t^2 g/h = \text{constant}$ or for $h = 0$, $tg/v_0 = \text{constant}$. Subsequently, by putting

$$R = R(t, u_0)$$

we obtain

$$u_0 t/R = \text{constant.}$$

[†] Even in undergraduate texts the student is not told why the mass is excluded as a variable.

Finally, for $v_0 = 0$, $R^2 g / u_0^2 h = $ constant or for $h = 0$, $Rg / u_0 v_0 = $ constant. Again the advocacy of directional characteristics is made unnecessary by a consideration of the analytical approximations implied in an answer. An example is that of the twist of a bar of rectangular cross-section due to the application of a torque. Expressing the angle of twist, α, as a function of the applied couple C, the cross-section dimensions a and b, the length l, the shear modulus G, and Young's modulus E, requires only the two dimensions of length and force. Then

$$\alpha = \alpha\left(\frac{C}{El^3}, \frac{a}{b}, \frac{a}{l}, \frac{G}{E}\right), \tag{18}$$

and this is the exact solution. It can be reduced further only by introducing approximations into the model of the phenomenon, and certainly not by dimensional analysis alone. These approximations are

1. Whilst still within the elastic range, large deflections are excluded so that $\alpha \propto C$.
2. There are no end effects inherent with large deflections, so that $\alpha \propto l$.
3. For small deflections, and when a/b is not very small, the problem becomes one of pure shear so that E can be excluded. This then reduces equation (18) to

$$\alpha G a^4 / Cl = f(a/b), \tag{19}$$

where the analytical elastic solution gives the function as an infinite series (Timoshenko and Goodier 1951).

There seems no justification for deriving equation (19) by assigning directional characteristics to certain variables instead of, as here, introducing the above approximations inherent in this result.

6. The choice and number of units conversion factors

Bridgman was really the first properly to face the difficulties over the use of universal physical constants in dimensional analysis, giving a stimulating discussion (Bridgman 1943) where others had so often ignored this problem. Later, general comments will be made upon primary quantities and universal constants. The following examples in this section are intended to supplement the discussion of § 3 and to lead to the general position.

In the problem of heating under forced convection by a compressible flow, all the equations are simultaneous, so that the independent variables appearing in the governing equations that control the velocity field become

$$q = f(u, \rho_0, \mu_0, l, C_v, R, M_0, \lambda_0, T_w, T_0) \tag{20}$$

where the second coefficient of viscosity, that appears in the stress equations for compressible flow (Sabersky and Acosta 1964), is taken to be directly proportional to μ. This results in the six non-dimensional groups

$$\frac{q}{U}, \frac{U^2 M_0}{T_0 R}, \frac{\rho_0 Ul}{\mu_0}, \frac{R}{M_0 C_v}, \frac{C_v \mu_0}{\lambda_0}, \frac{T_w}{T_0}.$$

These can be rearranged as

$$q/U, M_a, R_e, \gamma, P_r, T_w/T_0. \tag{21}$$

Alternatively the dependent group can be the Nusselt number, N_u, which is

$$N_u \equiv \dot{Q}/l^2(T_w - T_0).$$

Two questions arise from this example which appear not to have been posed before; the first concerns the omission of k_B from the list of variables in equation (20) despite it being the units conversion factor for temperature. In fact it does not appear in the equations as normally written. These contain R and M_0 in the combination R/M_0; however,

$$R/M_0 = k_B/m,$$

so that R could be replaced by k_B whilst M_0 would then be replaced by m. It would appear to be a less satisfactory formulation because, there being one less dimension required, there would be one extra non-dimensional group; however, this group is superfluous for the following reasons. It could be formulated as $(k_B\mu_0)/\lambda_0\rho_0 l^3)$. By substituting, from first-order kinetic theory of gases, the relation

$$\mu_0 = \rho_0 \bar{c} l_m$$

and using equation (14),

$$\frac{k_B\mu_0}{\lambda_0\rho_0 l^3} = \frac{1}{Nl^3}, \qquad (22)$$

which is a measure of a gas quantity. The total quantity of gas was excluded from the formulation of equation (20) for the reason given in a previous example. If it had been included, it could have been expressed as Nl_∞^3 and so the above group would then be expressible as l/l_∞; this is zero and so is excluded.

The second question also concerns the formulation of equation (20), in that it contains the six thermodynamic properties ρ_0, μ_0, C_v, M_0, k_0 and T_0. Thermodynamic texts (Keenan 1957) commonly say that a homogeneous system has a state that is a function of only two properties; Keenan expresses it by calling such a material a pure substance. But in fact a full description requires the statement of at least four properties; two, which are not a completely arbitrary choice, to fix the thermodynamic state, one to determine the chemical nature of the substance and one to define the amount of substance in the system. In some cases even a fifth might be required to specify the isotope. If account is to be taken of this in the present example so that only three properties are needed, then it might be concluded that three can be eliminated from equation (20) and so three groups would go from equation (21); knowledge of viscous compressible flow with heat transfer makes it clear that retention of only three groups would give an erroneous result: the answer now follows.

Accept firstly the proposition that each of C_v, λ_0, and T_0 is separately written as a function of ρ_0, μ_0, R/M_0 and k_B: it is necessary to combine R with the property M_0 in this way to enable cancellation of the quantity dimension, whilst k_B is used as the units conversion factor. Then the three relations lead to the three results

$$C_v M_0/R = \text{constant},$$

$$C_v \mu_0/\lambda_0 = \text{constant}$$

and

$$T_0^3 \rho_0^4 k_B^2 R/M_0\mu_0^6 = \text{constant}.$$

These three results are not universally valid for all gases, only specifically so in a way that is not inherent in the governing equations represented by equation (20). The usual specific form of the set of equation (20) ʳontains more than three properties; but those equations form an incomplete statement; the full set of equations would include ones such as $\mu = \mu(p, T, M_0, R)$.

The final answer then is that the statement of equation (20) contains a surplus of truly independent variables because it is really a set of simultaneous equations, solution of which would eliminate the superfluous thermodynamic properties; but no matter, dimensional analysis gives the desired form as equation (21). Further, equation (21) contains only the required number of three thermodynamic properties, which are

$$\gamma, P_r, lM_a/R_e,$$

the other quantities being boundary conditions.

Making an approximation to incompressible flow results, as for previous examples, in an uncoupling of the equations; this time the energy equation is uncoupled from the kinematic ones. Thus the velocity field is given by

$$q/U = f(\rho Ul/\mu_0),\tag{23}$$

in comparison with set (21). Then, for the temperature as the dependent variable, inspection of the energy equation and its boundary conditions shows

$$T = f(C_v, \mu_0/\rho, U, T_0, T_w, p/\rho, \lambda_0/\rho, l).\tag{24}$$

Now correspondingly to equation (23),

$$p/\rho U^2 = f(\rho Ul/\mu_0),$$

so that p/ρ is superfluous in equation (24). Also, as only temperature differences appear in the energy equation, equation (24) reduces to

$$(T - T_0) = f(C_v, \mu_0/\rho, U, (T_0 - T_w), \lambda_0/\rho, l).\tag{25}$$

The Π theorem reduces this to the four groups

$$\frac{T - T_w}{T_0 - T_w} = f\left(\frac{C_v\mu_0}{\lambda_0}, \frac{\rho Ul}{\mu_0}, \frac{\mu_0 U^2}{\lambda_0(T_0 - T_w)}\right).\tag{26}$$

A final approximation, commonly made, is to neglect the term in the energy equation represented by the dissipation function. This is found to be equivalent to neglecting the last group in equation (26); thus,

$$\frac{T - T_w}{T_0 - T_w} = f\left(\frac{C_v\mu_0}{\lambda_0}, \frac{\rho Ul}{\mu_0}\right).\tag{27}$$

The interesting feature of equation (25) is that the use of R/M_0 as a units conversion factor is not required because, as a result of the approximations made, the terms that remain in the energy equation each have a single dimension in temperature, so that effectively R/M_0 divides out. This is not so for equation (20).

This then is the basis of the Rayleigh–Raibouchinsky problem (Rayleigh 1915, Raibouchinsky 1915). Raibouchinsky was not entitled to put R/M_0 equal to unity, so as to express temperature in mechanical units, as R/M_0 had already been removed by the approximation inherent in equation (27). This is something more rigorous in its explanation than for instance Sedov's proposal that the units conversion factor J is

insignificant (Sedov 1959). Had Raibouchinsky tried his same procedure in the set of equation (20) he would have had no difficulty; the resulting number of non-dimensional groups would have been unchanged. It can be shown that the extra group that Raibouchinsky obtained is just an expression of the definition of density.

A further example, that of electrostatic streaming current, is directly comparable with that of forced heat convection. Again, because the electrical forces are so small, the mechanical and conservation of electric charge equations become uncoupled (Gibbings 1967). In that case, the units conversion factor F_a, the Faraday, is eliminated from the non-dimensional groups provided the boundary conditions are expressed analogously to those in the heat transfer case, that is in terms of a potential.

A simpler and well-discussed case analogous to that just considered is the one where the base units of work and heat are often set identical so that correspondingly the units conversion factor, J, is made dimensionless and put equal to unity. In dimensional analysis this results in the loss of a primary quantity, the unit of heat, and the corresponding loss of a variable, J, and so the number of dimensionless groups remains unchanged. But a requirement for doing this is that, in the problem, both mechanical and thermal effects must be accounted for.

7. Relation between primary quantities and units conversion factors

Several writers have in effect suggested that the number of primary quantities is one more than the number of primary units conversion factors, so that the dimensions of all quantities can be reduced to one primary quantity. In fact Buckingham suggested this in a little-known response (Buckingham 1915) to the Rayleigh–Raibouchinsky discussion previously mentioned. Later writers (e.g. Wilson 1942) have repeated the idea, though without reference to Buckingham.

The total number of primary quantities and hence, by the above principle, the total number of primary units conversion factors is, to some extent, arbitrary. For example, it has long been the practice to assign the value of unity to g_0 and J. It seems that the commonest form of present practice is to use the primary quantities

$$M, L, T, \theta, A, C, n, \alpha,$$

so that a set, totalling one less, of corresponding primary units conversion factors can be, respectively,

$$\gamma, h_p, k_B, \varepsilon_0, p_0, R, \beta_0.$$

It seems then that all other quantities that could also be regarded as units conversion factors can be expressed in terms of this set. Hence this particular selection of factors is an arbitrary one, though the total number is not, once the number of primary quantities has been chosen.

We now make the following proposal that, in part, has been illustrated by the previous examples. Philosophically, in any real phenomenon, to have absolute precision of understanding, all possible physical phenomena must be accounted for. This would require the use of all the above primary quantities and all the above primary units conversion factors.

This proposal can be amplified as follows. Forget, for a moment, practical considerations which require an approximate modelling of a real event in order to make it

tractable; but consider the most general position as follows:

1. Real events on a continuum scale must be influenced to some degree by all physical phenomena.
2. Therefore all primary quantities arising from the analytical expression of all these phenomena will appear in the variables governing the complete event.
3. Therefore all the primary units conversion factors associated with all the primary quantities will have to be included as variables.
4. The number of primary quantities is of arbitrary choice; the number of primary units conversion factors is one less.
5. The minimum number of primary quantities that can be used is one; the one to be used is of arbitrary choice.

(1) to (5) above imply the following corollary:

Suppose that another 'fundamental' relation—like for example Newton's law of gravitation, $F = \gamma m_1 m_2 / l^2$—is discovered, with its corresponding primary units conversion factor—in this example γ. Then either:

(i) we would have no primary quantity left for measurement; all that would be left to science would be counting;

or

(ii) we would have discovered a completely new primary quantity with its own unit of measurement and not related to any existing physical concepts except through its primary units conversion factor.

The practical study of a phenomenon requires, as already stated, approximate modelling, and hence not all physical phenomena are to be accounted for. Then only the appropriate quantities, and the units conversion factors corresponding to those that are primary quantities, will be included in the variables. The mode of procedure has been touched upon in the preceding examples and has been discussed at length elsewhere (e.g. Taylor 1974, Pankhurst 1971).

Thus the number of primary quantities and conversion factors to be used in a problem of dimensional analysis is not an arbitrary matter, but is strictly controlled by the approximations introduced in the modelling of the true physical event.

8. Listing of proposals

In the present paper, the following points are proposed as matters of basic principle in dimensional analysis.

(a) A distinction is made between the basic definition of a quantity and that of its measure, resulting, for example, in a definition of time that appears to resolve philosophical difficulties.

(b) Units conversion factors are laid down for angle and for amount of substance, and it is proposed that for the purposes of dimensional analysis these quantities are on a par with the primary quantities of the SI system.

(c) The importance of accounting for governing equations being uncoupled is demonstrated: this is proposed as a valid technique for use in dimensional analysis, to be preferred to the assigning of directional characteristics.

(d) Attention is drawn to the importance of recognising that many results in the literature that are purported to be derived using dimensional analysis rely in addition upon the imposition of approximations in the analysis.

(e) The thermodynamic limitation to the number of properties sufficient to define a

state is reconciled to the specification of a greater number for dimensional analysis.

(f) A resolution of the Rayleigh–Raibouchinsky problem is advanced.

(g) A statement of general principle is made concerning the number of primary quantities and the corresponding number of primary units conversion factors to be used in dimensional analysis.

Acknowledgments

Notation

Primary quantities:

M	mass		A	current
L	length		C	light intensity
T	time		n	amount
θ	temperature		α	angle

Symbol	Section	Meaning	Symbol	Section	Meaning
a	3	Area, amplitude	N	3, 6,	Concentration of
a, b	5	Cross-section dimensions			molecules
A, B	3, 5	Area, Vectors	N_a	3	Avogadro's number
A_0	3	Units conversion factor	N_u	6	Nusselt number
A_m	3	Unit atomic mass	p	3	Pressure
$\bar{c}$	3, 6	Mean velocity	p_0	4	Stream pressure
C	5	Couple	P_0	7	Mechanical equivalent of
C_v	6	Specific heats			light
d	4	Pipe diameter	Δp	4	Pressure drop
E	3, 5	Young's modulus	P_r	6	Prandtl number
F	3, 4, 5	Force	q	6	Velocity
F_A	6	Faraday constant	Q	4	Flow quantity
g	4, 5	Gravity acceleration	$\dot{Q}$	6	Heat rate
g_0	3	Units conversion factor	r	3	Radius
G	5	Modulus of rigidity	R	3, 5, 6, 7	Gas constant, range
h	5	Height	R_e	6	Reynolds number
h_p	7	Planck constant	s	3	Distance
J	6	Units conversion factor	t	3, 5	Time
k_B	3, 6, 7	Boltzmann constant	T	3	Tension, temperature
l	3, 4, 5, 6	Length	T_0	6	Boundary, stream
l_m	3, 6	Mean molecular path			temperature
l_∞	6	Size of flow	T_w	6	Wall temperature
L	4	Pipe length	U	4, 6	Velocity
m	3, 6	Mass, mass of molecule	U_0	4	Centre-line velocity
$\dot{m}$	4	Mass rate	u_0, v_0	5	Horizontal, vertical, initial
M_a	3, 6	Mach number			velocities
M_0	3, 6	Molecular mass	w	3	Load

Symbol	Section	Meaning	Symbol	Section	Meaning
α	3, 5	Angle	λ	3	Thermal conductivity
β	3	Angle, thermal expansion coefficient	λ_0	6	Stream thermal conductivity
β_0	3, 7	Units conversion factor for angle	μ	4	Viscosity
γ	3, 6, 7	Angle, gravitational constant, ratio of specific heats, deflection	μ_0	2, 6	Magnetic permeability of a vacuum, stream viscosity
			ρ	3, 4	Density
			ρ_0	6	Stream density
ε_0	7	Permittivity of space			

References

Augustine (Saint) 1961 *Confessions* Book II Transl. R S Pine-Coffin (London: Penguin) § 24

Bridgman P W 1943 *Dimensional Analysis* (New Haven: Yale)

Buckingham E 1915 *Nature* vol 96 p 39

Gibbings J C 1967 *Electrochim. Acta* vol 12 p 106

—— 1970 *Thermomechanics* (Oxford: Pergamon)

—— 1974 *The Experimental Method* ed. R K Penny (London: Longman) ch 3

Guggenheim E A 1942 *Phil. Mag.* vol 33 p 479

Huntley H E 1952 *Dimensional Analysis* (London: McDonal)

Jeffreys H 1943 *Phil. Mag.* vol 34 p 837

Keenan J H 1957 *Thermodynamics* (New York: Wiley)

Page C H 1967 *IEEE Trans.* E-10 p 70

Pankhurst R C 1971 *J. Franklin Inst.* vol 292 No 6 p 451

Plotinus 1968 *The Philosophy of Time* ed. R M Gale (London: Macmillan)

Raibouchinsky D 1915 *Nature* vol 95 p 591

Rayleigh (Lord) 1915 *Nature* vol 95 p 66

Sabersky R H and Acosta A J 1964 *Fluid Flow* (New York: Macmillan) pp 44, 61

Sedov L I 1959 *Similarity and Dimensional Methods in Mechanics* (London: Infosearch) p 43

Taylor E S 1974 *Dimensional Analysis for Engineers* (Oxford: Clarendon)

Timoshenko S P and Goodier J N 1951 *Theory of Elasticity* 3rd edn. (New York: McGraw-Hill) pp 312–3

Waismann F 1968 *The Philosophy of Time* ed. R M Gale (London: Macmillan)

Wilson W 1942 *Phil. Mag.* vol 33 p 32